Thermodynamics of fluid turbulence: A unified approach to the maximum transport properties

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Dissipative properties of various kinds of turbulent phenomena are investigated. Two expressions are derived for the rate of entropy increase due to thermal and viscous dissipation by turbulence, and for the rate of entropy increase in the surrounding system; both rates must be equal when the fluid system is in a steady state. Possibility is shown with these expressions that the steady-state properties of several different types of turbulent phenomena (Bénard-type thermal convection, turbulent shear flow, and the general circulation of the atmosphere and ocean) exhibit a unique state in which the rate of entropy increase in the surrounding system by the turbulent dissipation is at a maximum. The result suggests that the turbulent fluid system tends to be in a steady state with a distribution of eddies that produce the maximum rate of entropy increase in the nonequilibrium surroundings.

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I. INTRODUCTION

Turbulence seems to be one of the most frequent phenomena around us: thunderstorms in the atmosphere, turbulent eddies in the ocean, convection of a hot liquid in a coffee cup, shear turbulence of the liquid by forced mixing, etc. Yet there is no physical theory that is capable of expressing the complete dynamic structure of turbulence [1]. Understanding turbulence may be important because all living creatures, including human beings, are exposed to the turbulent motion of the atmosphere and ocean since the early beginning. Although the local turbulent motion is highly complex, it is hoped that the statistical-mean properties may have some general characteristics to be discovered. In this respect, such a statistical law of turbulence has been sought by numbers of investigators in various fields of physical sciences.

In the field of fluid dynamics, several suggestions have been made on maximum transport properties of turbulence on the basis of phenomenological observations. For thermal convection of a fluid layer heated from below (i.e., Bénard convection [2]), Malkus [3] suggested that the observed mean state represents a state in which the rate of heat transport by thermal convection is at a possible maximum ($F = \mathrm{Max.}$). For turbulent flow of a fluid layer under a pure shear, Malkus [4] and Busse [5] suggested that the realized state corresponds to a state with the maximum rate of momentum transport ($\tau = \mathrm{Max.}$). Their approach is now called the "optimum theory" or "upper bound theory," and is well known in the field [6–9]. However, the physical meaning of the maximized properties is yet to be determined.

A similar suggestion has been proposed in the field of earth science. Paltridge [10,11], for instance, suggested that the present mean state of the global atmosphere is reproducible as a state with a maximum rate of entropy increase due to heat transport by the general circulation of the atmosphere and ocean. Figure 1 shows such an example [10]. Without considering the detailed dynamics of the system, the predicted distributions (i.e., temperature, cloud amount, and horizontal heat flux) show considerably good agreement with observations. Later on, several researchers investigated his work, and obtained essentially the same result [12,13]. Although criticisms arose from a consideration of radiation entropy [14], it is recently shown that Paltridge's work remains valid if the rate of entropy increase by the turbulent heat transport is considered to be a maximum [15–17]. Thus, the global fluid system (the atmosphere and ocean) seems to be in a state with the maximum rate of entropy increase by the turbulent process ($\dot{S}_{\mathrm{turb}} = \mathrm{Max.}$), although the reason remains unclear. Moreover, until now, we do not have a reasonable explanation why the specific quantity (F, τ, or $\dot{S}_{\mathrm{turb}}$) tends to be maximized in each of these turbulent systems.

In order to clarify the issue in the above mentioned phenomena, we have investigated the dissipative properties of turbulence. In this paper, we shall not go into the details of the mathematical problems of the optimum theory; these are the subject of other considerations [5–9,18–21]. Instead, we shall present a simple thermodynamical proposition by which these apparently different types of turbulent phenomena (e.g., Bénard thermal convection, turbulent shear flow, and the general circulation of the global fluid system) can possibly be explained. The proposition states that a turbulent fluid system tends to be in a steady state with a maximum rate of entropy increase in the surrounding system by the turbulent dissipation in the fluid system. In what follows, we shall present a set of equations to express the rate of entropy increase in a fluid system and its surroundings (Sec. II). With these equations, we shall see how these apparently disparate turbulent phenomena can be explained by this simple proposition (Sec. III). Physical meaning of the proposition will shortly be discussed in Sec. IV.

II. ENTROPY INCREASE BY TURBULENT DISSIPATION

Let us consider the rate of entropy increase per unit time for a whole system consisting of a fluid system and its surrounding system with which the fluid system exchanges heat and momentum (Fig. 2). The rate of entropy increase due to

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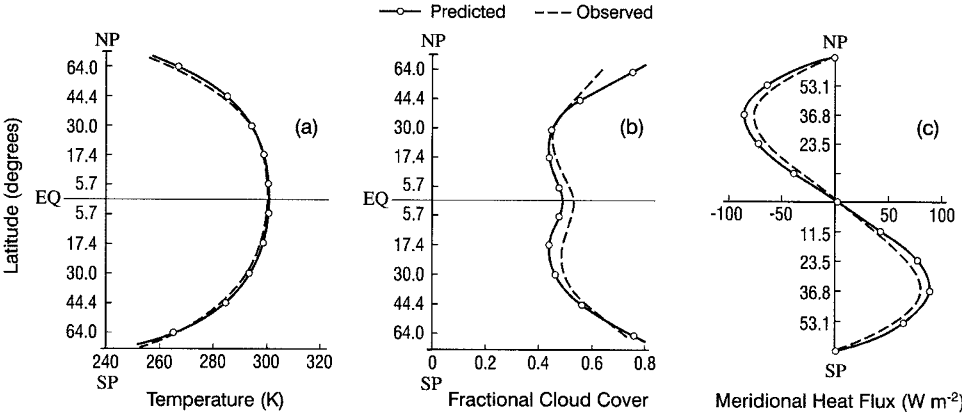

FIG. 1. Global distributions of: (a) mean air temperature, (b) cloud cover, and (c) horizontal heat transport in the earth. Solid line: predicted with $\dot{S}_{\mathrm{turb,st}}=\mathrm{Max.}$ and dashed line: observed (after Paltridge [10]).

some irreversible processes in the fluid system is then given by the sum of the contributions from each system [see Appendix Eq. (A7) as well as Ref. [17]] as

$$\dot{S}_{\mathrm{turb}}=\int\frac{1}{T}\left[\frac{\partial(\rho cT)}{\partial t}+\boldsymbol{\nabla}\cdot(\rho cT\mathbf{v})+p\boldsymbol{\nabla}\cdot\mathbf{v}\right]dV+\int\frac{F}{T}dA, \tag{1}$$

where ρ is the density of the fluid, c is the specific heat at constant volume, T is the absolute temperature, $\mathbf{v}$ is the velocity, p is the pressure, V is the volume of the fluid system, A is the surface bounding the system from the surroundings, and F is the diabatic heat flux due to turbulence at the boundary, defined as positive outwards. The first volume integral represents the entropy increase rate of the fluid system, and the second surface integral represents that of the surrounding system. If the concerned fluid system is in a steady state in a statistical sense, as usually the case of laboratory experiments, then the entropy, a state function of the fluid system, remains unchanged. In this case, Eq. (1) becomes simply

$$\dot{S}_{\mathrm{turb,st}}=\int\frac{F}{T}dA, \tag{2}$$

where the suffix st denotes that the fluid system is in a steady state. This equation suggests that the entropy produced by some irreversible processes in the turbulent fluid system is completely discharged into the surrounding system through the boundary heat flux F, so long as the fluid system is in a steady state. The entropy of the surrounding system is then increasing by the irreversible processes in the fluid system.[1] Previous studies [10–13,15–17] suggest that the increase rate by the turbulent dissipation processes tends to be a maximum ($\dot{S}_{\mathrm{turb,st}}=\mathrm{Max.}$) when the long-term mean state of the global fluid system is concerned (see Fig. 1 and Sec. III C).

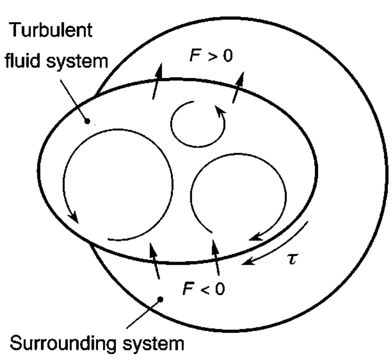

FIG. 2. A schematic representation of an open fluid system and its surrounding system with which the fluid system exchange heat F and momentum τ.

The general expression, Eq. (1), can be rewritten in a different form. It is known [22,23], and easy to show [Appendix Eq. (A10), [17]] that

$$\dot{S}_{\mathrm{turb}}=\int\mathbf{F}\cdot\boldsymbol{\nabla}\left(\frac{1}{T}\right)dV+\int\frac{\Phi}{T}dV, \tag{3}$$

where $\mathbf{F}$ is the diabatic heat flux density due to turbulence and Φ is the dissipation function, representing the rate of dissipation of kinetic energy into heat by viscosity per unit volume of the fluid. The first term is the rate of entropy increase by thermal dissipation, and the second term is that by viscous dissipation. The sum of the two terms represents the total rate of entropy increase by the turbulent dissipation. In a steady state, the entropy produced by the turbulent dissipation in the fluid system [Eq. (3)] is completely discharged into the surrounding system through the boundary heat flux [Eq. (2)]. If we assume, by analogy with the case of the global fluid system, that the turbulent fluid system tends to maximize the rate of entropy increase in the surrounding system by the turbulent dissipation, then we will get a proposition written in the following two different expressions:

$$\dot{S}_{\mathrm{turb,st}}=\int\mathbf{F}\cdot\boldsymbol{\nabla}\left(\frac{1}{T}\right)dV+\int\frac{\Phi}{T}dV \tag{4a}$$

$$=\int\frac{F}{T}dA=\mathrm{Maximum}. \tag{4b}$$

By using these two expressions [Eqs. 4(a) and 4(b)], it is possible to show that several maximum transport properties so far suggested for different types of turbulent phenomena can be explained with this proposition. For instance, the

[1] In this respect, the surrounding system is *not* in a steady state even though the fluid system is in a steady state (cf. [17]).

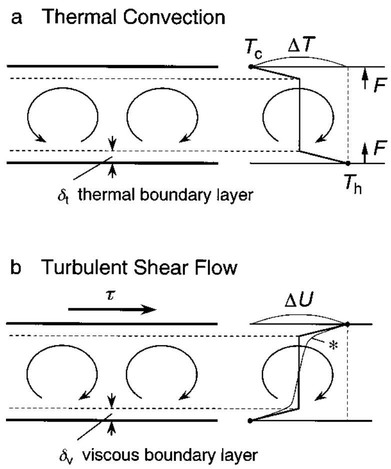

FIG. 3. Schematic illustrations of: (a) thermal convection and (b) turbulent shear flow.

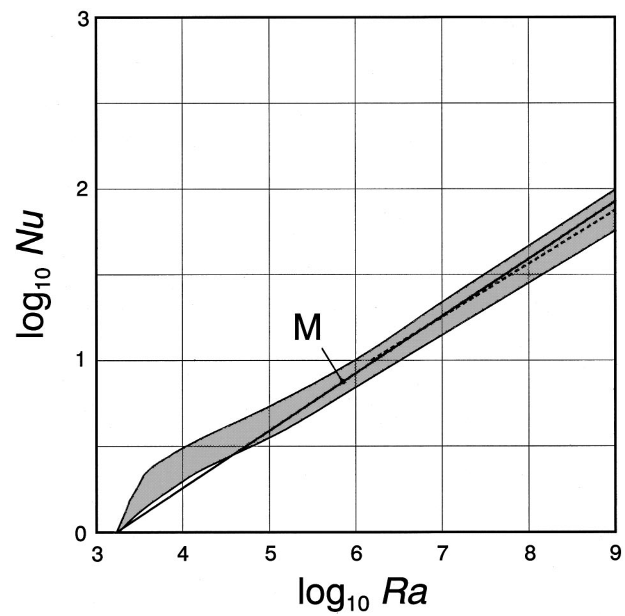

FIG. 4. Relation between the Nusselt number Nu and the Rayleigh number Ra. Solid line M: maximum estimate with Eq. (8) and shaded region: experimental results [7,25]. Dotted line shows recent experimental result [26].

maximum heat transport suggested for thermal convection [3] is consistent with Eq. (4b) to be a maximum. The maximum momentum transport (τ=Max.) suggested for turbulent shear flow [4,5] corresponds to the maximum shear working on the system, which results in the maximum viscous dissipation (Φ=Max.) in Eq. (4a). The maximum entropy increase suggested for the global fluid system [10–13,15–17] is identical to this proposition. In what follows, we shall discuss a few more details about these phenomena with respect to their specific boundary conditions.

III. MAXIMUM TRANSPORT PROPERTIES OF TURBULENCE

A. Thermal convection

Let us consider thermal convection of a fluid layer which is in contact with two thermal reservoirs with different temperatures; a hot reservoir (T_h) at the bottom, and a cold reservoir (T_c) at the top [Fig. 3(a)]. When the temperature difference $\Delta T = T_h - T_c$ becomes larger than a critical value determined by the Rayleigh number, the convective motion will start and develop [2,24]. If the fluid system can be considered to be in a statistically steady state, then the entropy of the system remains constant. In this case, the rate of entropy increase due to thermal convection is given by that in the surrounding system [Eq. (4b)]. The proposition of the maximum rate of entropy increase is then given by

$$\dot{S}_{\mathrm{turb,st}} = \int \frac{F}{T}\, dA = \frac{F}{T_c} - \frac{F}{T_h} = \frac{\Delta T F}{T_h T_c} = \mathrm{Max.} \tag{5}$$

Equation (5) shows that, provided that the boundary temperatures are kept constant, the proposition is identical to the maximum heat transport (F=Max.) suggested by Malkus [3].

As a simplest case, let us follow the boundary layer approach originated by Malkus [3]. Malkus suggested that the maximum F is attained by the largest temperature gradient at the thermal boundary layer (δ_t) adjacent to the boundary, where heat is mainly transported by thermal conduction. On the contrary, in the interior between the boundary layers, the convective heat transport by macroscopic eddies is so efficient that the temperature gradient in the interior is virtually negligible [Fig. 3(a)]. In this case, the maximum heat transport will be attained by the largest temperature gradient at the boundary layer with its minimum thickness ($\delta_{t,\mathrm{min}}$)

$$F_{\mathrm{max}} = k \frac{\Delta T/2}{\delta_{t,\mathrm{min}}}, \tag{6}$$

where k is the thermal conductivity. The minimum thickness $\delta_{t,\mathrm{min}}$ may be given by the stability criterion of Rayleigh (i.e., the threshold of a layer thickness above which convection would start [24]) as

$$\mathrm{Ra}^* = \frac{g \alpha \Delta T (2\delta_{t,\mathrm{min}})^3}{\kappa \nu}, \tag{7}$$

where Ra* is the critical Rayleigh number (~1700), g is the acceleration of gravity, and α, κ, and ν are the coefficients of volume expansion, thermal diffusivity, and kinematic viscosity, respectively. By substituting Eq. (7) in Eq. (6) and eliminating $\delta_{t,\mathrm{min}}$, one gets

$$F_{\mathrm{max}} = \frac{k \Delta T}{d} \left(\frac{\mathrm{Ra}}{\mathrm{Ra}^*} \right)^{1/3}, \tag{8}$$

where $\mathrm{Ra} \equiv g \alpha \Delta T d^3 (\kappa \nu)^{-1}$ is the Rayleigh number for the entire fluid layer (thickness d). It should be noted that Eq. (8) gives an upper bound for the heat transport that the boundary layer permits, in the sense that no dynamic constraint has been taken into account for the heat transport in the interior.

The maximum heat flux (line M) estimated by Eq. (8) and experimental results (shaded region) are shown in Fig. 4. The vertical axis is the dimensionless heat flux, the Nusselt number: $\mathrm{Nu} \equiv F(k\Delta T/d)^{-1}$, and the horizontal axis is the Rayleigh number, shown on a log-log plot. The experimental results are plotted from Refs. [7,25], and the result from recent experiment [26] is also shown by dotted line. The critical Rayleigh number is set to be $\mathrm{Ra}^* = 1708$ [25]. A reasonable agreement can be seen between the estimate and the experiments, despite of some discrepancy about the critical point ($\mathrm{Ra} \approx \mathrm{Ra}^*$). A slight overestimation of Eq. (8) can also be seen at large Rayleigh numbers; the scaling exponent of Eq. (8) is $1/3 \approx 0.33$, whereas that from the experiment [26] is 0.31. As mentioned above, the boundary layer approach gives an upper bound estimate for the heat flux without any dynamic constraint in the interior. Thus it may become invalid at the large Ra numbers [27–29]. The estimate can, in principle, be improved by adding additional possible constraints. For instance, Casting *et al.* [28] assumed a mixing zone adjacent to the boundary layer, and obtained a scaling exponent of $2/7 \approx 0.29$, while other attempts [19,27] suggested 0.5. Although the exact value of the scaling exponent is still debatable [26], the underlying conjecture on the maximum heat flux, and therefore on the maximum entropy increase, seems to be not unreasonable. In addition, results from recent numerical experiments of thermal convection of a rotating fluid system show that the system tends to select a regime with a higher rate of entropy increase [30]. These results suggest that a convective system tends to maximize the rate of entropy increase in the surrounding system [Eq. (4b)]. The maximum heat flux hitherto suggested for thermal convection can therefore be interpreted as a manifestation of Eq. (4) under the fixed temperature condition at the boundary.

B. Turbulent shear flow

Let us next consider turbulent shear flow of a fluid layer in contact with two reservoirs with different velocities; the relative velocity of the upper reservoir to the lower reservoir is ΔU [Fig. 3(b)]. When the relative velocity is larger than a certain critical value determined by the Reynolds number, the turbulent motion will start to develop [31]. In this case, the kinetic energy of the upper reservoir is transported into the fluid layer through the shear working at the upper boundary, and this energy is dissipated into heat by molecular diffusion in the fluid layer. In a steady state, the rate of viscous dissipation (viscous heating) must be balanced by the rate of working due to the shear stress τ times the relative velocity ΔU. The proposition of the maximum rate of entropy increase by the turbulent dissipation is then given by Eq. (4a) as

$$\dot{S}_{\mathrm{turb,st}} = \int \frac{\Phi}{T}\, dV \approx \frac{\Phi_t}{T} = \frac{\Delta U \tau}{T} = \mathrm{Max.}, \tag{9}$$

where $\Phi_t = \int \Phi\, dV$ is the total rate of viscous dissipation per unit surface of the fluid layer.[2] Equation (9) shows that, provided that the relative velocity is kept constant, the proposition is identical to the maximum shear stress (or, equivalently, maximum momentum transport) suggested by Malkus [4] and Busse [5].

As before, the maximum shear stress (or the maximum momentum transport) will be attained by the maximum velocity gradient at the viscous boundary layer adjacent to the boundary, whose minimum thickness $\delta_{v,\min}$ will be given by the stability criterion of Reynolds. In the interior between the boundary layers, the momentum transport by the turbulent eddies is so efficient that the velocity gradient may be virtually negligible [Fig. 3(b)]. With these assumptions justified, one will get

$$\tau_{\max} = \mu \frac{\Delta U/2}{\delta_{v,\min}} = \rho \frac{\Delta U^2}{\mathrm{Re}^*} = \frac{\mu \Delta U}{d} \frac{\mathrm{Re}}{\mathrm{Re}^*}, \tag{10}$$

where $\mu = \rho\nu$ is the viscosity, ρ is the density, $\mathrm{Re} \equiv d\Delta U/\nu$ is the Reynolds number for the entire fluid layer, and $\mathrm{Re}^* = 2\delta_{v,\min}\Delta U/\nu$ is the critical Reynolds number; above this threshold turbulence would occur [31]. Equation (10) has some important implications for the nature of homogeneous turbulence. For instance, a well-known empirical form of the surface drag stress is shown in Eq. (10) as $\tau_{\max} = C_D \rho \Delta U^2$, where $C_D = 1/\mathrm{Re}^*$ is the *drag coefficient*, empirically ranging from 0.001 to 0.01 [32]. A mean dissipation rate per unit mass of the fluid layer can be given by $\epsilon = \Phi_t/(\rho d) = \tau_{\max}\Delta U/(\rho d) = C_D \Delta U^3/d$; this shows a quantitative form of the Kolmogorov–Obukhov relation [22]. Further approach from Eq. (10) to a homogeneous turbulence theory is therefore promising, and will be dealt with in a separate paper.

The maximum shear stress (line M) estimated with Eq. (10) and experimental results (dots) are shown in Fig. 5. The vertical axis is a dimensionless shear stress, $\Gamma \equiv \tau(\mu \Delta U/d)^{-1}$, and the horizontal axis is the Reynolds number, shown on a log-log plot. The experimental results are plotted from Reichardt [33], and the critical Reynolds number is set to be $\mathrm{Re}^* \approx 500$ in reference to the experiment. Results from the Couette–Taylor flow experiment [34] are also shown with a dotted line, for a comparison. The agreement is again reasonable, despite some overestimation at large Reynolds numbers; the scaling exponent of Eq. (10) is 1, whereas that of Reichardt's experiment [33] is around 0.9. In the case of the Couette–Taylor experiment, the scaling exponent increases from 0.6 to 0.8 with increasing Re from 10^4 to 10^6 [34], suggesting that it asymptotically approaches 1 or 0.9 [18]. As before, Eq. (10) gives just an upper bound estimate for the momentum transport without any dynamic constraint in the interior. The estimate can thus be improved by taking into account additional constraints [9,21]. For ex-

[2] In this estimate, temperature is assumed to be almost uniform in the fluid layer. In a real steady state, the amount of viscous heating must be discharged into the surrounding system by thermal conduction through the boundary: $\Phi_t = \int F\, dA$. Thus, the expression Eq. (4b) is also valid in this case. But, it is impractical to estimate F by the small temperature gradient at the boundary.

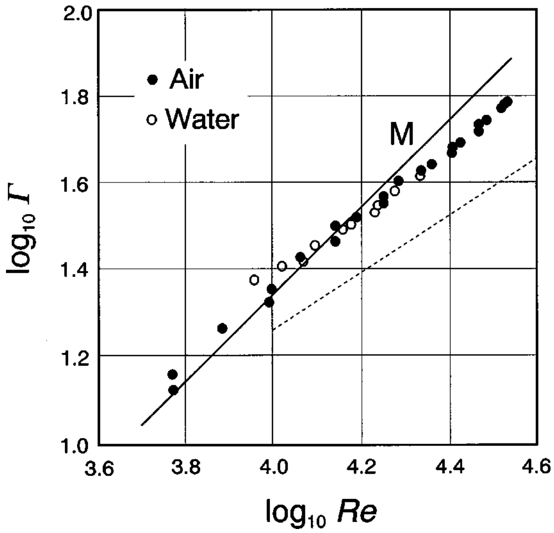

FIG. 5. Relation between the nondimensional shear stress, $\Gamma \equiv \tau(\mu \Delta U/d)^{-1}$, and the Reynolds number Re. Solid line M: maximum estimate with Eq. (10) and dots: laboratory experiment [33]. Dotted line shows results from Couette–Taylor experiment [34] for reference.

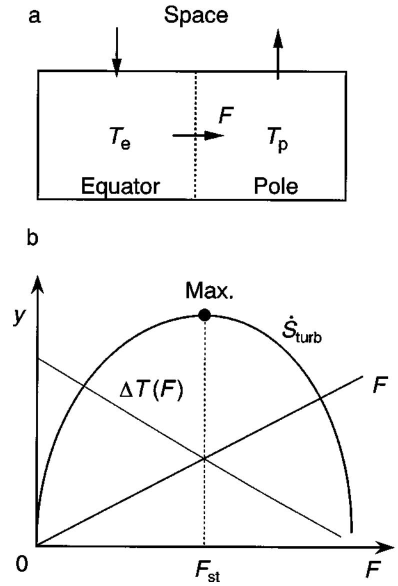

FIG. 6. (a) Schematic illustration of the earth consisting of two regions: equator and pole. F represents the horizontal heat transport by the circulation of the atmosphere and ocean. (b) Corresponding entropy increase rate in the surrounding system due to the heat transport, as a function of F. A maximum exists between the two extreme states: $F=0$ (no circulation) and $\Delta T(F)=0$ (extreme mixing).

ample, a rigorous analysis based on the dynamic equations and the continuity equation [5,6] suggests a velocity profile that is in qualitative agreement with the observed velocity profile [33], as shown in Fig. 3(b) with the asterisk. More detailed work is needed to improve the upper bound estimates for the momentum transport [20]. Here, it should be noted that the general agreements between the estimates and the experiments tend to support the proposition that the realized turbulent flow maximizes the shear stress, and therefore the rate of entropy increase [Eq. (4a)], under the fixed relative velocity condition at the boundary.

C. General circulation of the global fluid

The global fluid system of the earth (the atmosphere and ocean) is different from the convection system of a Bénard type, in the sense that the temperature difference at the boundary is not fixed but is a function of the heat transport itself. As a simplest case, let us consider the earth composed of two regions (the equator and pole); the average temperature in the equatorial region is T_e and that in the polar region is T_p [Fig. 6(a)]. In the present state, there is a net input of radiation (shortwave absorption–longwave emission) in the equatorial region, and a net output from the polar region. As a long-term mean state (steady state), this energy imbalance is compensated by energy transport F due to the direct motion of the atmosphere and ocean, called general circulation. Suppose an extreme case with no circulation (i.e., static state) with negligible amount of heat transport ($F \approx 0$). Then, the equatorial region will be heated up, and the polar regions will be cooled down. Because of the Stefan–Boltzmann law of radiation, this results in an increase in thermal emission from the equatorial region and a decrease from the polar region, thereby compensating the energy imbalance in each region. Thus, in the static state, the temperature difference will be the largest [Fig. 6(b)]. With increasing F from zero, the temperature difference will decrease. At very large F with extreme mixing, the temperature difference will become negligible. Thus, the temperature difference $\Delta T(F)=T_e-T_p$ is a monotonic decreasing function of F [Fig. 6(b)].

The rate of entropy increase by the heat transport F due to the general circulation is, provided that the fluid system is in a statistically steady state, given by the rate in the surrounding system [Eq. (4b)]. Then, the proposition is

$$\dot{S}_{\mathrm{turb,st}}=\frac{F}{T_p}-\frac{F}{T_e}=\frac{\Delta T(F)F}{T_eT_p}=\mathrm{Max}. \tag{11}$$

It should be noted that, unlike Eq. (5), the temperature difference $\Delta T(F)$ is not fixed but a decreasing function of F. Since $\dot{S}_{\mathrm{turb,st}}$ is proportional to the product of F and $\Delta T(F)$, it should have a maximum between the two extreme cases: $F=0$ (no circulation) and $\Delta T(F)=0$ (extreme mixing), as shown by the solid circle in Fig. 6(b). According to the proposition, this maximum corresponds to the most appropriate state for the general circulation. A number of attempts have been made to seek such a maximum in a more realistic system of the earth composed of 10–20 zones with different latitude and altitude [10–13,15]. Maxima were found in all these attempts, and the corresponding distributions of temperatures and heat fluxes show considerable agreements with the observations (see Fig. 1 as well as [10–13,15]). Thus, the general circulation seems to be regulated in a state with the *appropriate* rate of heat transport in the atmosphere and ocean, that produces the maximum rate of entropy increase in the surrounding system. This result shows a sharp contrast to the maximum heat transport found in Bénard-type thermal convection or the maximum momentum transport in turbulent shear flow. However, all these examples are completely in agreement with the proposition of the maximum entropy increase [Eq. (4)] presented here.

IV. DISCUSSION

We have seen in the preceding sections that the steady-state properties of several different types of turbulent phenomena (e.g., Bénard-type thermal convection, turbulent shear flow, and the general circulation of the global fluid) can be explained to a certain extent by a unique state of Eq. (4), i.e., a state with a maximum rate of entropy increases in the surrounding system by the turbulent dissipation in the fluid system. The appearance of the maximum transport properties of heat or momentum hitherto suggested for different turbulent systems is thereby interpreted as a manifestation of the same state of Eq. (4) under their specific boundary conditions. Equation (4) may thus be seen as a general thermodynamical tendency of various kinds of fluid turbulence, which can be added to a list of such important properties of turbulence as diffusivity, vorticity fluctuations, and dissipation (e.g., [35]). We do not in any way intend to say that our work is complete. More detailed work is needed in both fields: maximum transport theories of turbulence and maximum entropy output from the earth. However, the possible linkage of those apparently disparate turbulent phenomena suggests the existence of a basic law, which may also be of interest to scientists in various fields.

The idea that a turbulent fluid system (with large Ra or large Re) tends to produce a higher rate of dissipation is not very new. Almost a century ago, Terada and Hattori [36] carried out a series of careful experiments on turbulent motion of fluids, and pointed out that ‘‘the liquid has a *habit* of breaking up into a number of vortical portions (turbulent eddies), and such mode of motion is preferred by nature to the simpler laminar motion with less dissipation.’’ Félici [37] and Sawada [38] suggested that a convective system tends to select a regime of convection with a maximum rate of entropy increase due to the convective current. The former suggestion is related to the tendency of increased viscous dissipation rate (Φ), while the latter is related to the increased tendency of the convective current (F). These suggestions can also be incorporated into the single proposition of Eq. (4) presented here. Although the concept of the entropy increase similar to Eq. (4) has been hinted at already, to the best of our knowledge, an explicit statement of the proposition relating the maximum transport properties found both in the laboratory and the global fluid system to the state of maximum entropy increase has not been made before.

Finally, we shall discuss the reason why the turbulent fluid systems tend to maximize the rate of entropy increase in the surrounding system. A surrounding system consisting of reservoirs with a large difference in temperature or velocity is in a nonequilibrium state. If a small fluid system (size d) is in contact with such reservoirs, the corresponding Rayleigh or Reynolds number can be very large ($\mathrm{Ra} \gg \mathrm{Ra}^*$ or $\mathrm{Re} \gg \mathrm{Re}^*$). Then, the fluid system tends to reduce the nonequilibrium state by heat or momentum transport through the system, resulting in the entropy increase in the surrounding system. But, if the fluid system would contain no turbulent eddy at all (i.e., a static or laminar state), the heat or momentum should be transported only through molecular diffusion whose characteristic length scale (i.e., the mean free path) is very much smaller than the system size (d). Thus, the molecular diffusion in the static (or laminar) state is quite inefficient for the transport of heat or momentum. This static (or laminar) state is, however, unstable to small fluctuations when $\mathrm{Ra} \gg \mathrm{Ra}^*$ or $\mathrm{Re} \gg \mathrm{Re}^*$, and the fluctuations will develop into a train of turbulent eddies [36]. The development of these eddies will continue until they encounter the boundaries of the fluid system [Figs. 3(a) and 3(b)]. At this state, heat or momentum is transported most efficiently by the macroscopic eddies. We have seen in this paper that the transport is not only enhanced by the turbulent eddies, but also tends to maximize the rate of entropy increase in the surrounding system. This result suggests that the turbulent fluid system tends to be in a steady state with a distribution of eddies that produce the maximum rate of entropy increase in the surrounding system. In this sense, the initiation and evolution of turbulence seem to be regulated by a universal requirement of entropy increase in the nonequilibrium surroundings.

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APPENDIX

The time rate of change of entropy of an open fluid system is given by the following time derivative:

$$\dot{S}_{\mathrm{sys}} = \frac{d}{dt}\left[\int \rho s\, dV\right] = \int \frac{\partial(\rho s)}{\partial t} dV + \int \rho s v\, dA, \quad \text{(A1)}$$

where ρ is the density of the fluid, s is the entropy per unit mass, v is the normal component of fluid velocity at the surface (positive outward), V is the volume of the system, and A is the surface bounding the system. The first term on the right-hand side can be expanded and rewritten by using the equation of continuity $[\partial\rho/\partial t = -\nabla\cdot(\rho\mathbf{v})]$

$$\frac{\partial(\rho s)}{\partial t} = \rho\frac{\partial s}{\partial t} + s\frac{\partial\rho}{\partial t} = \rho\frac{\partial s}{\partial t} - \nabla\cdot(\rho s\mathbf{v}) + \rho\mathbf{v}\cdot\nabla s.$$

Substituting this in the volume integral of Eq. (A1), and transforming the second term by using Gauss's theorem, we get

$$\dot{S}_{\text{sys}}=\int \rho\left[\frac{\partial s}{\partial t}+\mathbf{v}\cdot\nabla s\right]dV. \tag{A2}$$

The expression in the square brackets is the substantial time derivative of entropy per unit mass of a fluid moving about in space (ds/dt). This rate of entropy increase can be expressed by using the thermodynamical relation $[ds\equiv\delta Q/T=\{du+pd(1/\rho)\}/T]$ as

$$\frac{ds}{dt}=\frac{1}{T}\left(\frac{du}{dt}+p\,\frac{d(1/\rho)}{dt}\right), \tag{A3}$$

where u is the internal energy per unit mass, and p is the pressure. Substituting this in Eq. (A2), and transforming the substantial time derivatives to spatial time derivatives $(du/dt=\partial u/\partial t+\mathbf{v}\cdot\nabla u)$, we get

$$\dot{S}_{\text{sys}}=\int \frac{1}{T}\left[\rho\,\frac{\partial u}{\partial t}+\rho\mathbf{v}\cdot\nabla u+p\nabla\cdot\mathbf{v}\right]dV. \tag{A4}$$

Here the continuity relation $[d(1/\rho)/dt=-(1/\rho^2)d\rho/dt=(1/\rho)\nabla\cdot\mathbf{v}]$ has been used. The first and second terms in the square brackets can be rewritten using the following relation:

$$\rho\,\frac{\partial u}{\partial t}+\rho\mathbf{v}\cdot\nabla u=\frac{\partial(\rho u)}{\partial t}+\nabla\cdot(\rho u\mathbf{v}).$$

By substituting it in Eq. (A4), we get

$$\dot{S}_{\text{sys}}=\int \frac{1}{T}\left[\frac{\partial(\rho cT)}{\partial t}+\nabla\cdot(\rho cT\mathbf{v})+p\nabla\cdot\mathbf{v}\right]dV. \tag{A5}$$

Here the relation $u=cT$ has been used, where c is the specific heat at constant volume. This equation is valid within the limits of an approximation that the temperature and the velocity are constant in the small volume element dV (cf. [22], Sec. 49).

Entropy of the surrounding system will increase by heat flux from the fluid system through the boundary. Following the definition of Clausius, the rate of entropy increase of the surrounding system is given by a surface integral of the heat flux due to turbulence divided by the temperature

$$\dot{S}_{\text{surr}}=\int \frac{F}{T}\,dA, \tag{A6}$$

where F is the surface heat flux defined as positive outward, dA is a small surface element, and the integral is taken over the whole boundary surface.

The rate of entropy increase due to turbulence for the whole system is given by the sum of Eqs. (A5) and (A6)

$$\dot{S}_{\text{turb}}=\int \frac{1}{T}\left[\frac{\partial(\rho cT)}{\partial t}+\nabla\cdot(\rho cT\mathbf{v})+p\nabla\cdot\mathbf{v}\right]dV+\int \frac{F}{T}\,dA. \tag{A7}$$

The first term represents the entropy increase rate of the open fluid system, and the second term represents that of the surrounding system.

The general expression Eq. (A7) can be rewritten in a different form. Because of the law of conservation of energy, the terms in the square brackets on the right-hand side of Eq. (A7) are related to the convergence of heat flux and the rate of heating by viscous dissipation (cf. [25], Sec. 7) as

$$\frac{\partial(\rho cT)}{\partial t}+\nabla\cdot(\rho cT\mathbf{v})+p\nabla\cdot\mathbf{v}=-\nabla\cdot\mathbf{F}+\Phi, \tag{A8}$$

where $\mathbf{F}$ is the diabatic heat flux density due to turbulence and Φ is the dissipation function, representing the rate of dissipation of kinetic energy by viscosity per unit volume of a fluid. The heat flux $\mathbf{F}$ includes all diabatic heat transport processes associated with turbulence, e.g., thermal conduction, latent heat transport by phase change, etc., but does not include the radiative transport process. The surface integral on the right-hand side of Eq. (A7) can be transformed to a volume integral by using Gauss's theorem

$$\int \frac{F}{T}\,dA=\int \frac{\nabla\cdot\mathbf{F}}{T}\,dV+\int \mathbf{F}\cdot\nabla\left(\frac{1}{T}\right)dV. \tag{A9}$$

By substituting Eqs. (A8) and (A9) in Eq. (A7), we get

$$\dot{S}_{\text{turb}}=\int \mathbf{F}\cdot\nabla\left(\frac{1}{T}\right)dV+\int \frac{\Phi}{T}\,dV. \tag{A10}$$

The first term is the entropy increase rate by thermal dissipation, and the second term is that by viscous dissipation.

[1] W. Heisenberg, Phys. Today **20(5)**, 27 (1967).
[2] H. Bénard, Ann. Chim. Phys. **23**, 62 (1901).
[3] W. V. R. Malkus, Proc. R. Soc. London, Ser. A **225**, 196 (1954).
[4] W. V. R. Malkus, J. Fluid Mech. **1**, 521 (1956).
[5] F. H. Busse, J. Fluid Mech. **41**, 219 (1970).
[6] F. H. Busse, Adv. Appl. Mech. **18**, 77 (1978).
[7] L. N. Howard, J. Fluid Mech. **17**, 405 (1963).
[8] L. N. Howard, Annu. Rev. Fluid Mech. **4**, 473 (1972).
[9] W. V. R. Malkus and L. M. Smith, J. Fluid Mech. **208**, 479 (1989).
[10] G. W. Paltridge, Q. J. R. Meteorol. Soc. **101**, 475 (1975).
[11] G. W. Paltridge, Nature (London) **279**, 630 (1979).
[12] H. Grassl, Q. J. R. Meteorol. Soc. **107**, 153 (1981).
[13] A. Noda and T. Tokioka, J. Meteorol. Soc. Jpn. **61**, 894 (1983).
[14] C. Essex, J. Atmos. Sci. **41**, 1985 (1984).
[15] H. Ozawa and A. Ohmura, J. Clim. **10**, 441 (1997).
[16] H. Ozawa, Phys. Rev. E **56**, 2811 (1997).

[17] S. Shimokawa and H. Ozawa, Tellus, Ser. A **53**, 266 (2001).

[18] C. R. Doering and P. Constantin, Phys. Rev. Lett. **69**, 1648 (1992); Phys. Rev. E **49**, 4087 (1994).

[19] C. R. Doering and P. Constantin, Phys. Rev. E **53**, 5957 (1996).

[20] R. R. Kerswell, Physica D **100**, 355 (1997); **121**, 175 (1998).

[21] R. R. Kerswell, Phys. Lett. A **272**, 230 (2000).

[22] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon, Oxford, 1987) [first published in Russian, 1944].

[23] S. R. de Groot and P. Mazur, *Non-equilibrium Thermodynamics* (North-Holland, Amsterdam, 1962).

[24] Lord Rayleigh, Philos. Mag. **32**, 529 (1916).

[25] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, Oxford, 1961).

[26] J. J. Niemela, L. Skrbek, K. R. Sreenivasan, and R. J. Donnelly, Nature (London) **404**, 837 (2000).

[27] R. H. Kraichnan, Phys. Fluids **5**, 1374 (1962).

[28] B. Castaing *et al.*, J. Fluid Mech. **204**, 1 (1989).

[29] S. Grossmann and D. Lohse, J. Fluid Mech. **407**, 27 (2000).

[30] S. Minobe, Y. Kanamoto, N. Okada, H. Ozawa, and M. Ikeda, Nagare, Multimedia **16** (2000) [http://www.nagare.or.jp/mm/2000/minobe/].

[31] O. Reynolds, Philos. Trans. R. Soc. London, Ser. A **186**, 123 (1894).

[32] C. H. B. Priestley, *Turbulent Transfer in the Lower Atmosphere* (University of Chicago Press, Chicago, 1959).

[33] H. Reichardt, Mitteilungen Max-Planck-Institut für Strömungsforschung und Aerodynamische Versuchsanstalt, Göttingen, Nr. 22, 1959.

[34] D. P. Lathrop, J. Fineberg, and H. L. Swinney, Phys. Rev. A **46**, 6390 (1992).

[35] H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT Press, Cambridge, 1972).

[36] T. Terada and K. Hattori, Rep. Aeronaut. Res. Inst. **2**, 87 (1926).

[37] N. J. Félici, C. R. Acad. Sci. Ser. B **278**, 151 (1974).

[38] Y. Sawada, Prog. Theor. Phys. **66**, 68 (1981).